كلية الهندسة بشبرا
FACULTY OF ENGINEERING AT SHOUBRA

Electromagnetic Fields / Fundamentals (ELE242)(CCE302)

Chapter (03)

Vectors Calculus & operators

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Vector Calculus

Del or Nabla Operator (∇ Operator)

- The del operator or Nabla Operator, written as ∇ is the vector differential operator. This vector differential operator otherwise known as the gradient operator is not a vector in itself, but when it operates on a scalar function, it produces a vector.
- The operator is useful in defining the following:

1. The gradient of a scalar V, written as ∇V.
2. The divergence of a vector $\vec{A}$, $\nabla.\vec{A}$
3. The curl of a vector $\vec{A}$, $\nabla x\vec{A}$
4. The Laplacian of a scalar V, $\nabla^2 V$

∇, the del operator vector in Cartesian *or* rectangular coordinates is defined by:

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \qquad (1.33a)$$

- ∇ in cylindrical coordinates is:

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z \qquad (1.33b)$$

- ∇ in spherical coordinates is:

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi \qquad (1.33c)$$

Vector Calculus

$$\nabla f(u_1,u_2,u_3) = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\bar{a}_{u_1} + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\bar{a}_{u_2} + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\bar{a}_{u_3}$$

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\bar{a}_x + \frac{\partial f}{\partial y}\bar{a}_y + \frac{\partial f}{\partial z}\bar{a}_z \qquad (1\text{-}42)$$

$\Rightarrow$ In cylindrical, $h_r = 1$, $h_\phi = r$, $h_z = 1$, so that:

$$\nabla f(r,\phi,z) = \frac{\partial f}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\bar{a}_\phi + \frac{\partial f}{\partial z}\bar{a}_z \qquad (1\text{-}43)$$

$\Rightarrow$ In spherical, $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$, so :

$$\nabla f(r,\theta,\phi) = \frac{\partial f}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\bar{a}_\phi \qquad (1\text{-}44)$$

Vector Calculus

⇒ Example 1-5 :

Determine the gradient of the following scalar fields:

a- $f(x, y, z) = xy^2 + 2z$

b- $f(r, \phi, z) = 2r \sin\phi$

c- $f(r, \theta, \phi) = 2\theta + r^2$

- Solution :

The gradient of f in the general form is :

$$\nabla f(u_1, u_2, u_3) = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\bar{a}_{u_1} + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\bar{a}_{u_2} + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\bar{a}_{u_3}$$

a- for rectangular ; $h_1 = h_2 = h_3 = 1$, thus :

$$\nabla f = \frac{\partial f}{\partial x}\bar{a}_x + \frac{\partial f}{\partial y}\bar{a}_y + \frac{\partial f}{\partial z}\bar{a}_z = y^2\bar{a}_x + 2xy\,\bar{a}_y + 2\,\bar{a}_z$$

b- for cylindrical; $h_1 = 1$, $h_2 = r$, $h_3 = 1$, thus :

$$\nabla f = \frac{\partial f}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\bar{a}_\phi + \frac{\partial f}{\partial z}\bar{a}_z = 2\sin\phi\,\bar{a}_r + 2\cos\phi\,\bar{a}_\phi$$

c- for spherical; $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$, thus :

$$\nabla f = \frac{\partial f}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\bar{a}_\phi = 2r\,\bar{a}_r + \frac{2}{r}\,\bar{a}_\theta$$

Grade (del , Nabla) (Sheet (03) - Prob (01))

(a) $f = 5x + 10xz - xy + 6$

$$\nabla f = \frac{\partial f}{\partial x}\bar{a}_x + \frac{\partial f}{\partial y}\bar{a}_y + \frac{\partial f}{\partial z}\bar{a}_z$$

$$\nabla f = (5 + 10z - y)\bar{a}_x + (-x)\,\bar{a}_y + (10x)\bar{a}_z$$

(b) $f = 2\sin\phi - rz + 4$

$$\nabla f = \frac{\partial f}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\bar{a}_\phi + \frac{\partial f}{\partial z}\bar{a}_z$$

$$\nabla f = (-z)\bar{a}_r + \frac{1}{r}(2\cos\phi)\bar{a}_\phi + (-r)\bar{a}_z$$

(c) $f = 2r\,\cos\theta - 5\phi + 2$

$$\nabla f = \frac{\partial f}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\bar{a}_\phi$$

$$\nabla f = 2\cos\theta\,\bar{a}_r + \frac{1}{r}(-2r\sin\theta)\bar{a}_\theta + \frac{1}{r\sin\theta}(-5)\bar{a}_\phi$$

$$\nabla f = 2\cos\theta\,\bar{a}_r - 2\sin\theta\,\bar{a}_\theta - \frac{5}{r\sin\theta}\,\bar{a}_\phi$$

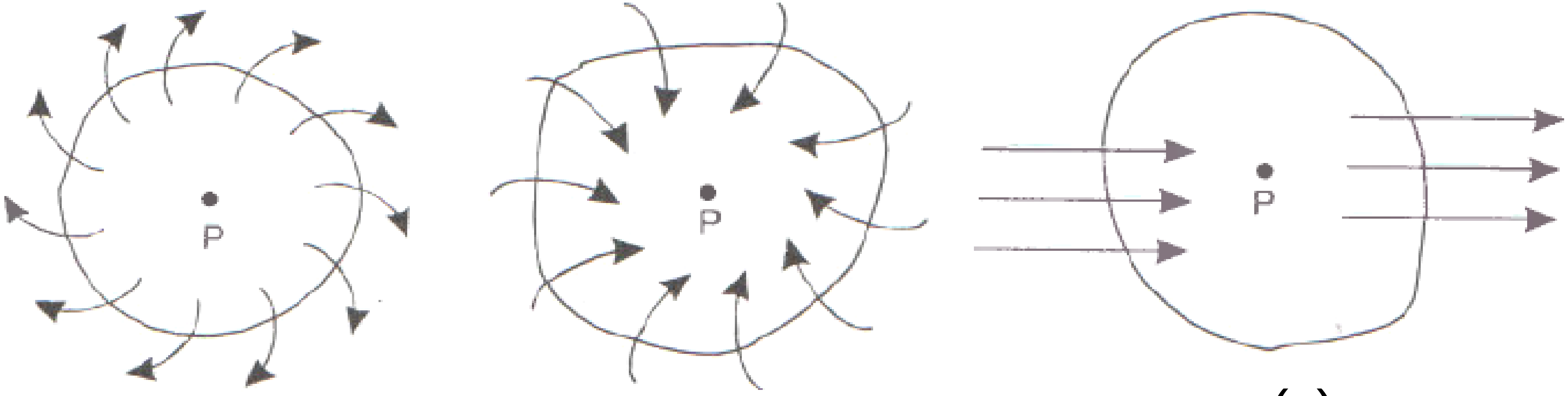
Vector Calculus (Continued)

1.5.3 Divergence of a Vector

- Divergence of a vector field is defined as the net outflow of flux per unit volume over a closed incremental surface.
- In other words, the divergence of $\vec{A}$ at a given point P is the outward flux per unit volume as the volume shrinks about P, as shown in Fig. 1.10 and defined as follows:

$$Div\vec{A} = \nabla.\vec{A} \qquad Div\vec{A} = \nabla.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad (1.35a)$$

where ΔV is the volume enclosed by the closed surface S in which P is located.



(a) (b) (c)

Divergence of a vector(a); Positive divergence; (b) Negative divergence; (c) Zero divergence

DIV

$$div\ \bar{F} = \left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right)\cdot\left(F_x\bar{a}_x + F_y\bar{a}_y + F_z\bar{a}_z\right)$$

Thus: $div\bar{F} = \nabla\cdot\bar{F}$

For other coordinate systems, equation (1-71) will serve only a symbolic role.

⇒ In general orthogonal coordinate systems :

$$\nabla\cdot\bar{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial u_1} + \frac{\partial(h_1h_3F_2)}{\partial u_2} + \frac{\partial(h_1h_2F_3)}{\partial u_3}\right]$$

⇒ Thus, in cylindrical coordinate system :

$(u_1, u_2, u_3) = (r, \phi, z)$, and $h_1 = 1$, $h_2 = r$, $h_3 = 1$

$$\nabla\cdot F(r,\phi,z) = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}$$

⇒ For a spherical coordinate system :

$(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$

$$\nabla\cdot\bar{F}(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F) + \frac{1}{r\sin\theta}\frac{\partial F_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

DIV (Sheet (03) – Prob. (02))

Determine the divergence of the following fields :

$(a)\ \bar{A} = x^2\bar{a}_x + yz\ \bar{a}_y + xy\ \bar{a}_z$

$(b)\ \bar{A} = r\sin\phi\ \bar{a}_r + 2r\cos\phi\ \ \bar{a}_\emptyset + 2z^2\ \bar{a}_z$

$(c)\ \bar{A} = 5\sin\theta\ \bar{a}_\theta + 5\sin\phi\ \bar{a}_\phi$ at $\left(0.5, \frac{\pi}{4}, \frac{\pi}{4}\right)$

$(a)\ \bar{A} = x^2\bar{a}_x + yz\ \bar{a}_y + xy\ \bar{a}_z$

$$\nabla.\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla.\bar{A} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xy)$$

$$\boxed{\nabla.\bar{A} = 2x + z}$$

$(b)\ \bar{A} = r\sin\phi\ \bar{a}_r + 2r\cos\phi\ \ \bar{a}_\emptyset + 2z^2\ \bar{a}_z$

$$\nabla.\bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla.\bar{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(r(r\sin\phi)\right) + \frac{1}{r}\frac{\partial}{\partial \phi}(2r\cos\phi) + \frac{\partial}{\partial z}(2z^2)$$

$$\nabla.\bar{A} = \frac{1}{r}\ [2r\ \sin\phi] + [-2\sin\phi] + 4z$$

$$\boxed{\nabla.\bar{A} = 4z}$$

DIV (Sheet (03) – Prob. (02))

Determine the divergence of the following fields :

$(a)\ \bar{A} = x^2\bar{a}_x + yz\ \bar{a}_y + xy\ \bar{a}_z$

$(b)\ \bar{A} = r\sin\phi\,\bar{a}_r + 2r\cos\phi\ \bar{a}_\emptyset + 2z^2\ \bar{a}_z$

$(c)\ \bar{A} = 5\sin\theta\,\bar{a}_\theta + 5\sin\phi\ \bar{a}_\phi$ at $\left(0.5,\frac{\pi}{4},\frac{\pi}{4}\right)$

$(c)\ \bar{A} = 5\sin\theta\,\bar{a}_\theta + 5\sin\phi\ \bar{a}_\phi$ at $\left(0.5,\frac{\pi}{4},\frac{\pi}{4}\right)$

$$\nabla.\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla.\bar{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,(5\sin\theta)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(5\sin\phi)$$

$$\nabla.\bar{A} = \frac{1}{r\sin\theta}\left(5(2\ \sin\theta\cos\theta)\right) + \frac{1}{r\sin\theta}(5\cos\phi)$$

$$\boxed{\nabla.\bar{A} = \frac{5}{r\sin\theta}(\sin 2\theta + \cos\phi)}$$

$$\nabla.\bar{A}\Big|_{\left(0.5,\frac{\pi}{4},\frac{\pi}{4}\right)} = \frac{5}{0.5\ \text{x}\ \sin\frac{\pi}{4}}\left(\sin\frac{\pi}{2} + \cos\frac{\pi}{4}\right)$$

$$\nabla.\bar{A}\Big|_{\left(0.5,\frac{\pi}{4},\frac{\pi}{4}\right)} = 24.142$$

DIV (Sheet (03) – Prob. (03))

Show that the vector field $\bar{F} = e^{-y}(\cos x\, \bar{a}_x - \sin x\, \bar{a}_y)$ solenoidal

Answer

$$\nabla . \bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla . \bar{F} = -\sin x\, e^{-y} + \sin x\, e^{-y}$$

$$\because \nabla . \bar{F} = 0$$

$\therefore$ $\bar{F}$ is solenoidal

There is neither source nor sink

DIV (Sheet (03) – Prob. (04))

If the electric field $\bar{E} = y\bar{a}_x + x\bar{a}_y$, show that the given region does not contain any electric charge .

Answer

$$\nabla . \bar{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla . \bar{E} = 0$$

$$\because \nabla . \bar{E} = 0$$

$\therefore$ There is neither source nor sink

i.e. There is no electrical charge

Vector Calculus (Continued)

Curl of a Vector

- Curl of a vector field is defined as an axial (or rotational) vector whose magnitude is the maximum circulation of per unit area as the area tends to zero and the direction is the normal direction of the area, as shown in Fig. 1.11 and defined as follows:
- Physically, the curl of a vector field represents the rate of change of field strength in a direction at right angles to the field and is a measure of rotation of something in a small volume surrounding a particular point. For streamline motions, the curl is zero, while it is maximum near eddies and whirlpools. (How much Flow of field)
- The vector fields whose curl is zero are called irrotational. Irrotational fields are also called as conservative fields.
- In Cartesian system, the curl of vector field can be found using:

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{a}_1 & h_2 \mathbf{a}_2 & h_3 \mathbf{a}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

$$Curl\vec{A} = \nabla x\vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

CURL (Sheet (03) – Prob. (05))

Compute the curl of the following vector fields

a) $\overline{F} = xy\bar{a}_x + 2yz\bar{a}_y - \bar{a}_z$

b) $\overline{F} = 2\bar{a}_r + \sin\phi\,\bar{a}_\phi - z\bar{a}_z$

c) $\overline{F} = r\bar{a}_r + \bar{a}_\theta + \sin\theta\,\bar{a}_\phi$

a) $\overline{F} = xy\bar{a}_x + 2yz\bar{a}_y - \bar{a}_z$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & -1 \end{vmatrix} = -2y\bar{a}_x - x\bar{a}_z$$

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b) $\overline{F} = 2\bar{a}_r + \sin\phi\,\bar{a}_\phi - z\bar{a}_z$

$$\nabla \times \bar{F} = \frac{1}{\rho}\begin{vmatrix} \bar{a}_\rho & \rho\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \bar{a}_\rho & \rho\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 2 & \rho\sin\phi & -z \end{vmatrix} = \frac{1}{\rho}\sin\phi\ \bar{a}_z$$

CURL (Sheet (03) – Prob. (05))

Compute the curl of the following vector fields

a) $\bar{F} = xy\bar{a}_x + 2yz\bar{a}_y - \bar{a}_z$

b) $\bar{F} = 2\bar{a}_r + \sin\phi\,\bar{a}_\phi - z\bar{a}_z$

c) $\bar{F} = r\bar{a}_r + \bar{a}_\theta + \sin\theta\,\bar{a}_\phi$

c) $\bar{F} = r\bar{a}_r + \bar{a}_\theta + \sin\theta\,\bar{a}_\phi$

$$\nabla \times \bar{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta\,A_\phi \end{vmatrix} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ r & r & r\sin^2\theta \end{vmatrix}$$

$$\nabla \times \bar{F} = \frac{1}{r^2 \sin\theta}\left[\bar{a}_r(2r\sin\theta\cos\theta) - r\bar{a}_\theta(\sin^2\theta) + r\sin\theta\,\bar{a}_\phi(1)\right]$$

$$\nabla \times \bar{F} = \frac{2\cos\theta}{r}\bar{a}_r - \frac{\sin\theta}{r}\bar{a}_\theta + \frac{1}{r}\bar{a}_\phi$$

1.4 Vector Integrals

The familiar concept of integration will now be extended to cases when the integrand involves a vector.

1.4.1 Line Integral

A line means the path along a curve in space. Terms such as line, curve and contour can also be used interchangeably. Given a vector field $\vec{A}$ and a curve L, the integral

$$\int_L \vec{A}.d\vec{l} = \int_a^b |A| \cos\theta \, |dl| \qquad (1.28)$$

is defined as the line integral of $\vec{A}$ around L, It is the tangential component of along the curve L. If the path of integration is a closed curve such as abca as in Fig. 1.8, then the integral becomes a closed contour integral, which is called circulation of $\vec{A}$ around L.

$$\oint_L \vec{A}.d\vec{l} \qquad (1.29)$$

1.4 Vector Integrals (Continued)

1.4.2 Surface Integral

Consider Fig. 1.9. A vector field $\vec{A}$ continuous in a region containing the smooth surface S, is given. The surface integral or flux of $\vec{A}$ through S is defined as

$$\Psi = \int_S \vec{A}.d\vec{S} = \int_S \vec{A}.\vec{a}_n dS = \int_S |A| \cos\theta dS \qquad (1.30)$$

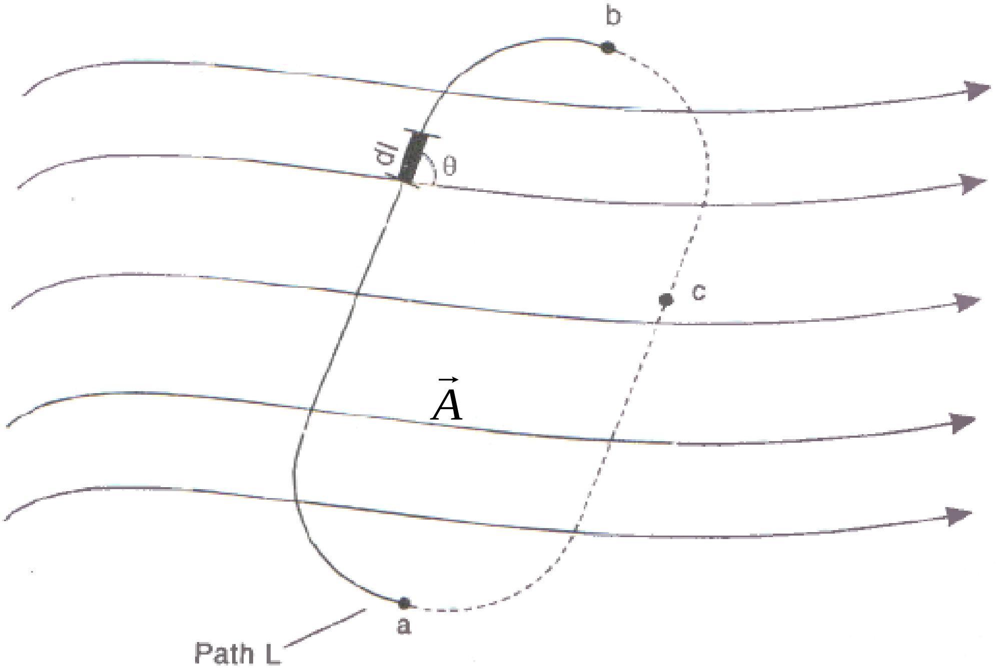


Fig. 1.8 Line Integral.

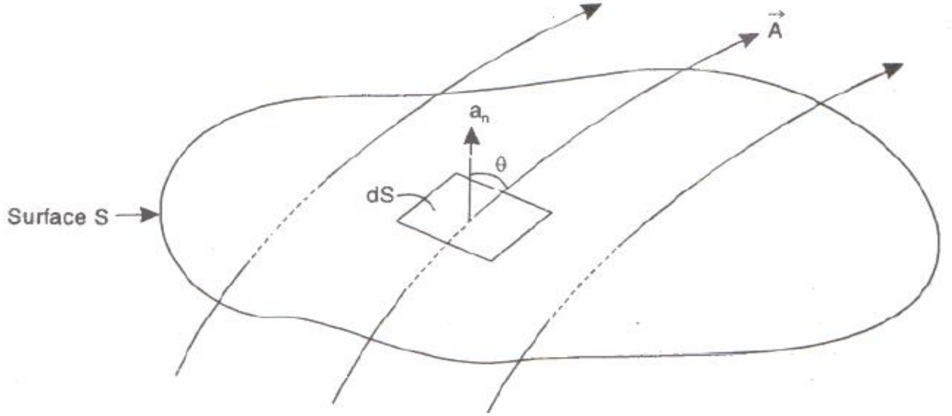


Fig. 1.9 Surface Integral.

1.4 Vector Integrals (Continued)

1.4.2 Surface Integral

where $\hat{a}_n$ is the unit vector normal to S at any point. For a closed surface, the equation becomes:

$$\Psi = \oint_S \vec{A}.d\vec{S} \qquad (1.31)$$

- which is referred to as the net outward flux of $\vec{A}$ from S. Notice that a closed path defines an open surface whereas a closed surface defines a volume.

1.4.3 Volume Integral

The volume integral is defined as the volume integral of scalar ρ_v over the volume v as:

$$\int_V \rho_v dv \qquad (1.32)$$

1.5 Vector Calculus (Continued)

1.5.5 Laplacian Operation

- The Laplacian of a scalar field V can be written as ∇^2V. It is defined as the divergence of the gradient of V.
- In Cartesian coordinates system, ∇^2 (V is a scalar field is a $\vec{A}$ vector field is:

$$\nabla^2 V = \nabla . \nabla V = \nabla . gradV \qquad (1.37a)$$

$$\nabla^2 \vec{H} = \nabla(\nabla . \vec{H}) - \nabla x \nabla x \vec{H} \qquad (1.37b)$$

$$\nabla^2 V = \nabla . \nabla V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \qquad (1.37c)$$

Properties of Laplacian Operation

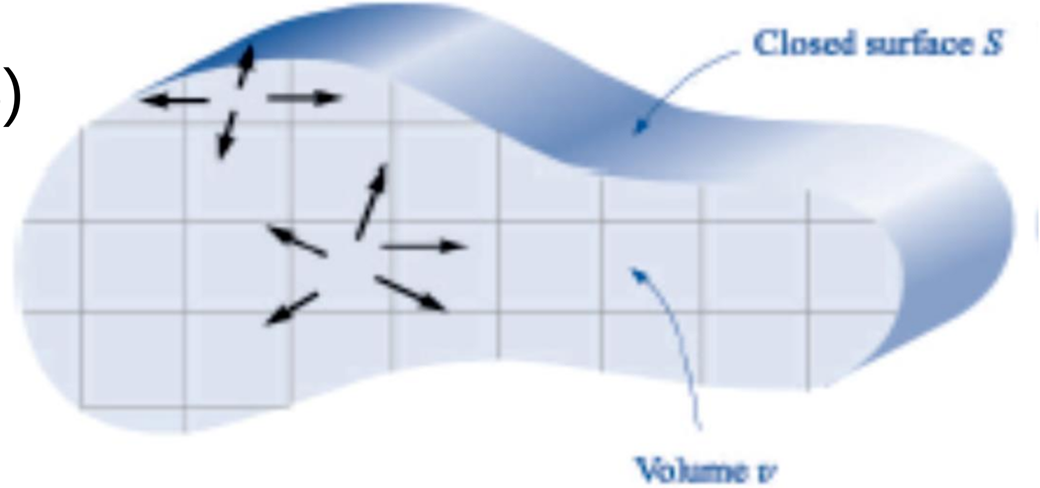
1. Laplacian of a scalar field is another scalar field.
2. If Laplacian of a scalar field is zero in a given region, then the scalar field is said to be harmonic in that region.

1.5 Vector Calculus (Continued)

1.5.6 The Divergence Theorem

- This theorem applies to any vector field for which the appropriate partial derivatives exist. It states that the integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.
- Its benefits drive from the fact that it relates a triple integration throughout some volume to double integration over the surface of that volume as shown in Fig. 1.12 and stated by the following expression:

$$\oint_{S} \vec{A}.d\vec{S} = \int_{vol} \nabla.\vec{A}.dv \qquad (1.38)$$



(Fig. 1.12)

Divergence theorem states that the integral of the normal component of a vector function over a closed surface is equal to the integral of the divergence of that vector throughout the volume 'v' enclosed by the surface ‘S'.

1.5 Vector Calculus (Continued)

1.5.7 Stokes' Theorem

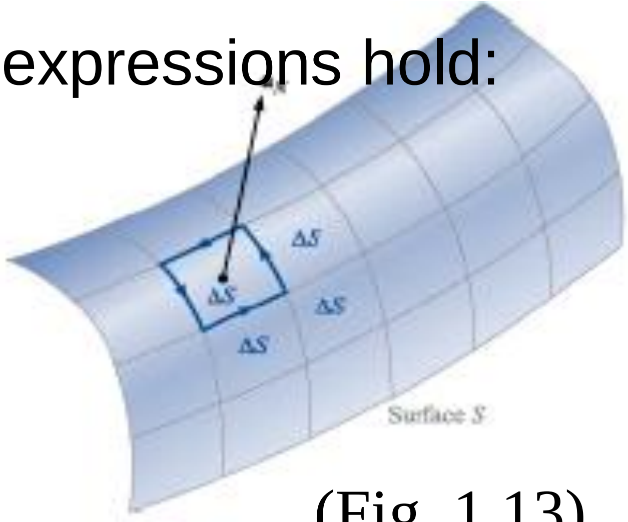
- The line integral of the tangential component of a vector $\vec{A}$ around a closed path C is equal to the surface integral of the normal component of curl $\vec{A}$ over the surface enclosed by the path C.
- This theorem applies to any vector field. It relates a surface integral a closed line integral shown in Fig.1.13 and stated by the following expression:

$$\oint_C \vec{A}.d\vec{l} = \int_S (\nabla x \vec{A}).d\vec{s} \qquad (1.39)$$

1.5.8 Grad, Div and Curl Identities

- For any scalar or vector field the following expressions hold:

$$\begin{aligned} &\nabla.\nabla x\vec{A} = 0 \\ &\nabla x(\nabla V) = 0 \qquad (1.40) \\ &\nabla x \nabla x \overleftarrow{A} = \nabla(\nabla.\vec{A}) - \nabla^2 \vec{A} \end{aligned}$$



(Fig. 1.13)

Prob (06)

Prove that curl grad ϕ= 0, where ϕ is a scalar field.

Solution Curl grad ϕ = ∇ x Vϕ is given by:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{a}_x + \frac{\partial\phi}{\partial y}\vec{a}_y + \frac{\partial\phi}{\partial z}\vec{a}_z \qquad \nabla\times\nabla\phi = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$= \left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial y\,\partial z}\right)\vec{a}_x - \vec{a}_y\left(\frac{\partial^2\phi}{\partial x\,\partial z} - \frac{\partial^2\phi}{\partial x\,\partial z}\right) + \left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial x\,\partial y}\right)\vec{a}_z = 0.$$

Prob (07)

Determine the Divergence and Curl of the following vector field.

$$\vec{P} = x^2 yz\hat{a}_x + xz\hat{a}_z$$

Solution

The Divergence of is given by:

$$\nabla \cdot \vec{P} = \frac{\partial}{\partial x}(P_x) + \frac{\partial}{\partial y}(P_y) + \frac{\partial}{\partial z}(P_z) \;= 2xyz + 0 + x \;= 2xyz + x.$$

The Curl of is given by:

$$\nabla \times \vec{P} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 yz & 0 & xz \end{vmatrix}$$

$$= (0)\vec{a}_x - \vec{a}_y\left(z - x^2 y\right) + \vec{a}_z\left(x^2 z\right)$$

$$= \left(x^2 y - z\right)\vec{a}_y + x^2 z\vec{a}_z .$$

Prob (08)

Find the scalar component of $\vec{A}$ along $\vec{B}$.

$$\vec{A} = 6\hat{a}_x + 2\hat{a}_y - 3\hat{a}_z \qquad \text{and} \qquad \vec{B} = 3\hat{a}_x - 4\hat{a}_y$$

Solution

The component of $\vec{A}$ along $\vec{B}$ is given by:

$$A_B = \left(\vec{A} \cdot \vec{a}_B\right)$$

$$\vec{a}_B = \frac{\vec{B}}{\left|\vec{B}\right|} = \frac{3\vec{a}_x - 4\vec{a}_y}{\sqrt{3^2 + (-4)^2}}$$

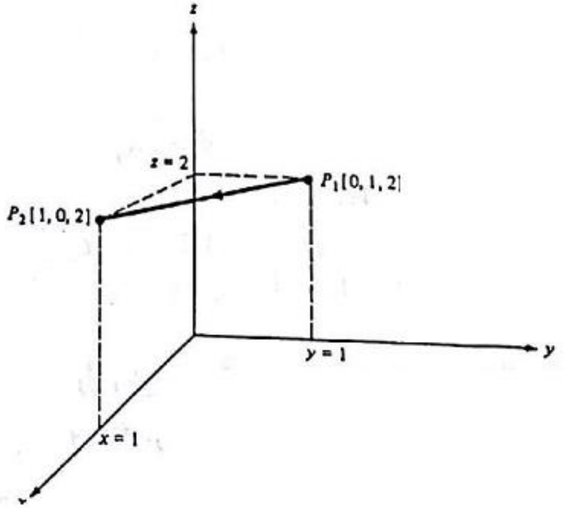
$$A_B = \left(\vec{A} \cdot \vec{a}_B\right) = (6, 2, -3) \cdot \left(\frac{3}{5}, \frac{-4}{5}, 0\right) = \frac{18}{5} - \frac{8}{5} = 2 .$$

Prob (09)

Evaluate the line integral of the vector field

$$\bar{F}(x,y,z)=(x+y)\,\bar{a}_x - x\,\bar{a}_y + z\,\bar{a}_z$$

along the path of straight line from $P_1\,[0,1,2]$ to $P_2\,[1,0,2]$, as shown in



- Solution :

The line integral of $\bar{F}$ becomes :

$$\int_{P_1}^{P_2} \bar{F}\cdot d\bar{\ell} = \int_{x=0}^{1} (x+y)\,dx - \int_{y=1}^{0} x\,dy + \int_{z=2}^{2} z\,\cancel{dz}_{\to=0}$$

The equation of the path is the equation of st. line that can be determined as : $\left(\frac{y-0}{x-1} = \frac{0-1}{1-0}\right)$

$$y = 1 - x$$

Substituting this relation yields

$$\int_{P_1}^{P_2} \bar{F}\cdot d\bar{\ell} = \int_{x=0}^{1} dx - \int_{y=1}^{0} (1-y)\,dy$$

$$= x\Big|_0^1 - \left(y - \frac{y^2}{2}\right)\Big|_1^0 = \frac{3}{2}$$

Prob (10)

Consider the scalar field f :

$$f(x, y, z) = 2xy + 3$$

Prove that the vector field $\bar{F} = \nabla f$ is a conservative field for the closed contour of paths c_1, c_2, and c_3 shown in figure

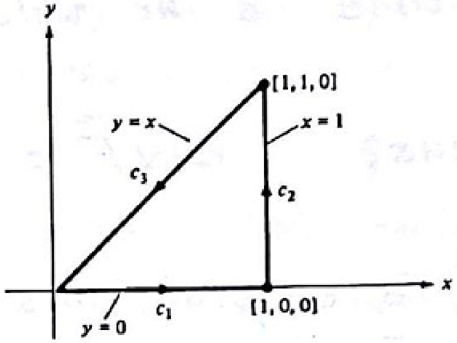


Fig. 1-15

- Solution :

To prove that the vector field $\bar{F}$ is conservative, i.e. it is required to prove that

$$\oint_c \bar{F} \cdot d\bar{\ell} = 0$$

We have :

$$\bar{F} = \nabla f = \frac{\partial f}{\partial x} \bar{a}_x + \frac{\partial f}{\partial y} \bar{a}_y + \frac{\partial f}{\partial z} \bar{a}_z$$

$$= 2y \, \bar{a}_x + 2x \, \bar{a}_y$$

$$d\bar{\ell} = dx \, \bar{a}_x + dy \, \bar{a}_y + dz \, \bar{a}_z$$

$$\therefore \bar{F} \cdot d\bar{\ell} = 2y \, dx + 2x \, dy$$

Prob (11)

$$\therefore \oint_{C_1} \bar{F}\cdot d\bar{\ell} = \int_{C_1} \bar{F}\cdot d\bar{\ell} + \int_{C_2} \bar{F}\cdot d\bar{\ell} + \int_{C_3} \bar{F}\cdot d\bar{\ell}$$

$\Rightarrow$ For C_1 : $\displaystyle\int_{[0,0,0]}^{[1,0,0]} \bar{F}\cdot d\bar{\ell} = \int_{x=0}^{x=1} 2\underset{\to y=0}{\cancel{y}}\,dx = 0$ (I) $\Rightarrow$

becuse for C_1 :
$y=0,\ z=0$
$dy=0\ \ dz=0$
$x=0\to 1$

$\Rightarrow$ For C_2 : $\displaystyle\int_{[1,0,0]}^{[1,1,0]} \bar{F}\cdot d\bar{\ell} = \int_{y=0}^{y=1} 2\underset{\to x=1}{\cancel{x}}\,dy = 2$ (II) $\Rightarrow$ because for C_2 :
$x=1\ \ z=0\ \ \ dx=0\ \ dz=0$
$y=0\to 1$

$\Rightarrow$ For C_3 : $\displaystyle\int_{[1,1,0]}^{[0,0,0]} \bar{F}\cdot d\bar{\ell} = \int_{x=1}^{x=0} 2\underset{\to y=x}{\cancel{y}}\,dx + \int_{y=1}^{y=0} 2\underset{\to y=x}{\cancel{x}}\,dy \Rightarrow$

$$= x^2\Big|_1^0 + y^2\Big|_1^0 = -2 \quad \text{(III)}$$

because for C_3 :
$y=x\ \ \ z=0$
$x=1\to 0\ \ \ y=1\to 0$

From I, II, III, it is clearly that $\oint_C \bar{F}\cdot d\bar{\ell} = 0$

i.e. the field is conservative.

Thank you for your attention

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